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2002 J. Phys. A: Math. Gen. 35 747

(http://iopscience.iop.org/0305-4470/35/3/318)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 747–757 PII: S0305-4470(02)27214-5

The Darboux-like transform and some integrable cases of the q-Riccati equation

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Received 20 July 2001, in final form 24 October 2001
Published 11 January 2002
Online at stacks.iop.org/JPhysA/35/747

Abstract
Using the q-version of the Darboux transform we obtain the general solution of q-difference Riccati equation from a special one by the action of one-parameter group. This allows us to construct the solutions for the large class of q-difference Riccati equations as well as q-difference Schrödinger equations, which are different from those obtained by the standard Darboux transform.

PACS numbers: 02.30.Uu, 02.30.Ik, 02.10.De, 02.30.Gp

Introduction

In this paper we investigate the Darboux-like factorization method for the q-difference version of Riccati and Schrödinger equations. It appears that this method, which is by all means effective for differential Riccati and Schrödinger equations [D, C, I-H, L-R, M-S, Mil, N-D], leads to non-trivial and interesting results in the q-deformed case too. Some of the new formulae have their undeformed versions. They tend at the limit $q \to 1$ to the limits which are well known in differential case.

The Darboux factorization

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} + u(x)\right)\left(-\frac{\mathrm{d}}{\mathrm{d}x} + u(x)\right) \tag{1}$$

gives the well-known correspondence between the one-dimensional Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)\psi(x) = 0 \tag{2}$$

and the Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x) = -u^2(x) + V(x) \tag{3}$$

where

$$u(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi(x). \tag{4}$$

This correspondence is a starting point for a search of exact solutions of both the equations above [M-S].

We will investigate, in this paper, an analogue of the Darboux method for the pair of q-difference equations

$$\partial_q\begin{pmatrix}\psi(x)\\ \varphi(x)\end{pmatrix} = \begin{pmatrix}R(x) & S(x)\\ V(x) & T(x)\end{pmatrix}\begin{pmatrix}\psi(x)\\ \varphi(x)\end{pmatrix} \tag{5}$$

and

$$V(x) = \partial_q u(x) - T(x)u(x) + R(x)u(qx) + S(x)u(x)u(qx) \tag{6}$$

whose solutions are related by

$$u(x) = \frac{\varphi(x)}{\psi(x)}. \tag{7}$$

It is clear that (5) and (6) generalize (2) and (3) respectively. The Schrödinger and Riccati equations are obtained in the limit $q \to 1$ under the additional assumption that $R(x) = 0 = T(x)$.

Let us recall here that the q-derivative and q-integral are defined by

$$\partial_q\psi(x) = \frac{\psi(x) - \psi(qx)}{(1-q)x} \tag{8}$$

$$\int_0^x \psi(t)\,\mathrm{d}_q t = \sum_{n=0}^{\infty}(1-q)q^n x\psi(q^n x) \tag{9}$$

respectively, where $0 \leqslant q \leqslant 1$. The standard derivative and integral are obtained for $q = 1$. However, the reason for the investigation of the q-difference equations (5) and (6) is not only that they generalize in a natural way the Schrödinger and Riccati equations.

If one additionally assumes $1 - (1-q)xT(x) = 0$ in the real case and one takes q^n instead of the real argument x, equation (5) appears to reduce to the three-term recurrence equation

$$\psi_{n+2} = [1 - (1-q)q^{n+1}R(q^{n+1})]\psi_{n+1} + (1-q)^2 q^{2n+1}S(q^{n+1})V(q^n)\psi_n \tag{10}$$

for the function $\psi_n := \psi(q^n)$ of the natural argument $n \in \mathbb{N} \cup \{0\}$. Hence, the q-difference equation (5) can be applied to those physical problems which are related to the theory of orthogonal polynomials [Su].

The paper is organized in the following way. In section 1 we introduce the q-difference Darboux transform and integrate equation (5) for the case in which $V(x) = 0$. The action of the q-difference Darboux transform is presented in section 2. There we find a one-parameter auto-Bäcklund transform for the q-difference Riccati equation and show that it generates the general solution of (6) from a special one. Section 3 is devoted to the presentation of some extended classes of solutions of the q-difference Schrödinger and Riccati equation. All results presented in sections 1 and 2 have well-known differential counterparts and this aspect is also exhibited in the paper. Finally, we present solutions for the families of Riccati equations which are different from those obtained by the Darboux transform. It shows that the method proposed in the paper is similar but not the same as the Darboux factorization method.

1. The q-difference Darboux-like transform

In order to solve the q-difference equation (5) by the iterative method we will rewrite it in the following form:

$$\begin{pmatrix}\psi(qx)\\ \varphi(qx)\end{pmatrix} = \Lambda(x)\begin{pmatrix}\psi(x)\\ \varphi(x)\end{pmatrix} \tag{11}$$

where

$$\Lambda(x) = \mathbf{I} - (1-q)x\begin{pmatrix}R(x) & S(x)\\ V(x) & T(x)\end{pmatrix}. \tag{12}$$

Let us assume here that $R(x)$, $S(x)$, $V(x)$ and $T(x)$ are continuous functions of a real argument. Hence, the matrix sequence

$$\Lambda(q^{n-1}x)\cdots\Lambda(qx)\Lambda(x) =: \Lambda(x;q)_n \tag{13}$$

is pointwise convergent,

$$\Lambda(x;q)_n \to_{n\to\infty} \Lambda(x;q)_\infty \tag{14}$$

to a matrix function $\Lambda(x;q)_\infty$. The inverse matrix function $\Lambda(x;q)_\infty^{-1}$ is exactly the resolvent of equation (5), i.e.

$$\begin{pmatrix}\psi(x)\\ \varphi(x)\end{pmatrix} = \Lambda(x;q)_\infty^{-1}\begin{pmatrix}\psi(0)\\ \varphi(0)\end{pmatrix}. \tag{15}$$

So the problem of solving (5) is equivalent to the calculation of the infinite matrix product

$$\Lambda(x;q)_\infty := \prod_{n=0}^{\infty}\Lambda(q^n x). \tag{16}$$

The above suggests the following transform:

$$\Lambda(x) \to D(qx)^{-1}\Lambda(x)D(x) = \Lambda'(x) \tag{17}$$

$$\begin{pmatrix}\psi(x)\\ \varphi(x)\end{pmatrix} \to D(x)^{-1}\begin{pmatrix}\psi(x)\\ \varphi(x)\end{pmatrix} = \begin{pmatrix}\psi'(x)\\ \varphi'(x)\end{pmatrix} \tag{18}$$

where $D(x)$ is a $GL(2,\mathbb{R})$-valued function of the real argument. It is obvious that the transform (17), (18) preserves the form of equation (11) and the transformed resolvent $\Lambda'(x;q)_\infty^{-1}$ is related to the initial one by

$$\Lambda'(x;q)_\infty^{-1} = D(x)^{-1}\Lambda(x;q)_\infty^{-1}D(0). \tag{19}$$

Thus the virtue of the above transform is to find such a matrix-valued function $D(x)$ for equation (5), which reduces the unknown resolvent $\Lambda(x;q)_\infty^{-1}$ to some known one $\Lambda'(x;q)_\infty^{-1}$.

We will find later the explicit form of the resolvent $\Lambda'(x;q)_\infty$ in the case when $V'(x) = 0$. So, in order to integrate (5) it is enough to transform (16) to the upper triangular matrix function $\Lambda'(x)$ by the use of (17), (18). Any matrix can be decomposed generically into the product of upper triangular and lower triangular matrices. Thus, without loss of generality, we can assume that

$$D(x) = \begin{pmatrix}1 & 0\\ c(x) & 1\end{pmatrix}. \tag{20}$$

After substituting (20) into (17) we find that $\Lambda'(x)$ will be an upper triangular matrix if and only if the function $c(x)$ satisfies the q-difference Riccati equation (6).

The q-difference Schrödinger operator factorizes into the form

$$-\partial_q^2 + V(x) = (\partial_q + u(qx))(-\partial_q + u(x)) \tag{21}$$

iff the function $u(x)$ satisfies equation (6) with $R(x) = T(x) = 0$ and $S(x) = 1$. Hence, it is natural to call the matrix transform (17), (18) the *q-difference Darboux transform*.

We will use the identity

$$\prod_{n=0}^{\infty}(1-(1-q)q^n x f(q^n x)) = \exp\left(\frac{1}{1-q}\int_0^x \frac{\ln(1-(1-q)tf(t))}{t}\,\mathrm{d}_q t\right) \tag{22}$$

which is an easy consequence of the definition of the q-integral.

Proposition 1. *If $R'(x)$, $S'(x)$ and $T'(x)$ are continuous functions and $V'(x) = 0$, then*

$$\Lambda'(x;q)_\infty = \begin{bmatrix} \exp\frac{1}{1-q}\int_0^x \frac{\ln(1-(1-q)tR'(t))}{t}\,\mathrm{d}_q t & B(x) \\ 0 & \exp\frac{1}{1-q}\int_0^x \frac{\ln(1-(1-q)tT'(t))}{t}\,\mathrm{d}_q t \end{bmatrix} \tag{23}$$

where

$$\begin{aligned} B(x) = \left[-\exp\left(\frac{1}{1-q}\int_0^x \frac{\ln(1-(1-q)tT'(t))}{t}\,\mathrm{d}_q t\right)\right]\left[\int_0^x \frac{S'(t)}{1-(1-q)tR'(t)}\right. \\ \left.\times\exp\left(\frac{1}{1-q}\int_0^t \frac{1}{s}\ln\frac{1-(1-q)sR'(s)}{1-(1-q)sT'(s)}\,\mathrm{d}_q s\right)\mathrm{d}_q t\right]. \end{aligned} \tag{24}$$

Proof. Since $V'(x) = 0$ we can assume that $\Lambda'(x;q)_\infty$ is the upper triangular matrix of the form

$$\Lambda'(x;q)_\infty = \begin{bmatrix} \prod_{n=0}^{\infty}(1-(1-q)q^n xR'(q^n x)) & B(x) \\ 0 & \prod_{n=0}^{\infty}(1-(1-q)q^n xT'(q^n x)) \end{bmatrix}. \tag{25}$$

Then from the equation

$$\Lambda'(x;q)_\infty = \Lambda'(qx;q)_\infty\Lambda'(x) \tag{26}$$

we find that the function $B(x)$ does satisfy

$$B(x) = [1-(1-q)xT'(x)]B(qx) - \prod_{n=0}^{\infty}(1-(1-q)q^{n+1}xR'(q^{n+1}x))(1-q)xS'(x). \tag{27}$$

Equation (27) is solved by the iterative method. One finally gets

$$\begin{aligned} B(x) = -\prod_{n=0}^{\infty}(1-(1-q)q^n xT'(q^n x))\sum_{n=0}^{\infty}\frac{(1-q)q^n xS'(q^n x)}{1-(1-q)q^n xR'(q^n x)} \\ \times\prod_{k=n}^{\infty}\frac{(1-(1-q)q^k xR'(q^k x))}{(1-(1-q)q^k xT'(q^k x))}. \end{aligned} \tag{28}$$

Substituting (28) into (25) and using identity (22), we obtain the formulae (23) and (24). □

2. The solution of the q-difference Schrödinger equation and auto-Bäcklund transform for the q-difference Riccati equation

We have obtained in section 1 the resolvent function $\Lambda'(x;q)_\infty^{-1}$ for the case of $V'(x) = 0$ (proposition 1). Let us stick to this case and let $\Lambda'(x)$ of (17) remain the upper triangular matrix function. Applying the q-difference Darboux-like transform to $\Lambda(x)$, with $D(x)$ given by (20), we find the general solution of the q-differential equation (5).

Proposition 2. *Let us assume that functions $R(x)$, $S(x)$, $T(x)$ and $V(x)$ from equations (5) are continuous. Let $(\psi_0(x) \quad \varphi_0(x))^\top$ be a particular solution of (5) and*

$$u_0(x) = \frac{\varphi_0(x)}{\psi_0(x)}. \tag{29}$$

Then the potential $V(x)$ is given by

$$V(x) = \partial_q u_0(x) - T(x)u_0(x) + R(x)u_0(qx) + S(x)u_0(x)u_0(qx) \tag{30}$$

and the general solutions of (5) are given by

$$\begin{aligned}\psi(x) = \exp&\left(-\frac{1}{1-q}\int_0^x \frac{\ln\{1-(1-q)t[R(t)+u_0(t)S(t)]\}}{t}\,\mathrm{d}_q t\right)\\ &\times\left[D + F\int_0^x \frac{S(t)}{1-(1-q)t[R(t)+u_0(t)S(t)]}\right.\\ &\left.\times\exp\left(\frac{1}{1-q}\int_0^t \frac{1}{s}\ln\frac{1-(1-q)s[R(s)+u_0(s)S(s)]}{1-(1-q)s[T(s)-u_0(qs)S(s)]}\,\mathrm{d}_q s\right)\mathrm{d}_q t\right]\end{aligned} \tag{31}$$

$$\begin{aligned}\varphi(x) = F\exp&\left(-\frac{1}{1-q}\int_0^x \frac{\ln\{1-(1-q)t[T(t)-u_0(qt)S(t)]\}}{t}\,\mathrm{d}_q t\right)\\ &+ u_0(x)\exp\left(-\frac{1}{1-q}\int_0^x \frac{\ln\{1-(1-q)t[R(t)+u_0(t)S(t)]\}}{t}\,\mathrm{d}_q t\right)\\ &\times\left[D + F\int_0^x \frac{S(t)}{1-(1-q)t[R(t)+u_0(t)S(t)]}\right.\\ &\left.\times\exp\left(\frac{1}{1-q}\int_0^t \frac{1}{s}\ln\frac{1-(1-q)s[R(s)+u_0(s)S(s)]}{1-(1-q)s[T(s)-u_0(qs)S(s)]}\,\mathrm{d}_q s\right)\mathrm{d}_q t\right]\end{aligned} \tag{32}$$

The constants D and F are related to the initial conditions by:

$$\begin{aligned} D &= \psi(0)\\ F &= -\psi(0)\frac{\varphi_0(0)}{\psi_0(0)} + \varphi(0).\end{aligned} \tag{33}$$

Proof. In order to prove the formulae (31), (32) and (33) we assume in (12) that $\Lambda'(x)$ is upper triangular and apply the q-difference Darboux-like transform (17) with

$$D(x) = \begin{pmatrix} 1 & 0\\ u_0(x) & 1\end{pmatrix}. \tag{34}$$

Formula (30) follows now from (17) and from the property of $\Lambda'(x)$ of being upper triangular. Formulae (31) and (32) are obtained from proposition 1 and (33). □

As a corollary of proposition 2 we obtain the auto-Bäcklund transform for the q-difference Riccati equation.

Proposition 3. *Let $u_0(x)$ be some special solution of equation (6). Then the general solution of (6) is given by*

$$\begin{aligned}u^t(x) &= \left(\mathcal{B}_t^+ u_0\right)(x) = u_0(x)\\ &+ \frac{t\exp\left(\frac{1}{1-q}\int_0^x \frac{1}{y}\ln\frac{1-(1-q)y[R(y)+u_0(y)S(y)]}{1-(1-q)y[T(y)-u_0(qy)S(y)]}\,\mathrm{d}_q y\right)}{1 + t\int_0^x \frac{S(y)}{1-(1-q)y[R(y)+u_0(y)S(y)]}\exp\left(\frac{1}{1-q}\int_0^y \frac{1}{s}\ln\frac{1-(1-q)s[R(s)+u_0(s)S(s)]}{1-(1-q)s[T(s)-u_0(qs)S(s)]}\,\mathrm{d}_q s\right)\mathrm{d}_q y}\end{aligned} \tag{35}$$

where $t \in \mathbb{R}$.

Proof. Formula (35) is obtained by substituting (31) and (32) into (7) and putting $t = \frac{F}{D}$. □

The q-difference Schrödinger equation

$$\left(-\partial_q^2 + V(x)\right)\psi(x) = 0 \tag{36}$$

is a special case of (5) and is obtained by putting $R(x) = T(x) = 0$ and $S(x) = 1$. From proposition 2 we may draw the following:

Corollary 1. *The solution of the q-difference Schrödinger equation with the potential*

$$V(x) = \partial_q u_0(x) + u_0(x)u_0(qx) \tag{37}$$

is given by

$$\psi(x) = \exp\left(-\frac{1}{1-q}\int_0^x \frac{\ln(1-(1-q)tu_0(t))}{t}\,\mathrm{d}_q t\right)\left[D + F\int_0^x \frac{1}{1-(1-q)tu_0(t)}\right.$$
$$\left.\times \exp\left(\frac{1}{1-q}\int_0^t \frac{1}{s}\ln\frac{1-(1-q)su_0(s)}{1+(1-q)su_0(qs)}\,\mathrm{d}_q s\right)\mathrm{d}_q t\right]. \tag{38}$$

In the limit of $q \to 1$ the q-difference equations (5) and (6) tend to their differential counterparts

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi(x)\\ \varphi(x)\end{pmatrix} = \begin{pmatrix}R(x) & S(x)\\ V(x) & T(x)\end{pmatrix}\begin{pmatrix}\psi(x)\\ \varphi(x)\end{pmatrix} \tag{39}$$

and

$$V(x) = \frac{\mathrm{d}}{\mathrm{d}x}u(x) + (R(x) - T(x))u(x) + S(x)u^2(x) \tag{40}$$

where $u(x)$ is given by (7). Propositions 2 and 3 are valid in the limit $q \to 1$ too. Therefore, we can apply them to the differential case and reproduce, in such a way, the formulae for general solutions of (39) and (40). They are given by

$$\psi(x) = \exp\left(\int_0^x (R(t) + u_0(t)S(t))\,\mathrm{d}t\right)$$
$$\times\left[D + F\int_0^x S(t)\exp\left(\int_0^t [T(s) - R(s) - 2u_0(s)S(s)]\,\mathrm{d}s\right)\mathrm{d}t\right] \tag{41}$$

$$\varphi(x) = F\exp\left(\int_0^x (T(t) - u_0(t)S(t))\,\mathrm{d}t\right) + u_0(x)\exp\left(\int_0^x (R(t) + u_0(t)S(t))\,\mathrm{d}t\right)$$
$$\times\left[D + F\int_0^x S(t)\exp\left(\int_0^t [T(s) - R(s) - 2u_0(s)S(s)]\,\mathrm{d}s\right)\mathrm{d}t\right] \tag{42}$$

and

$$u^t(x) = u_0(x) + \frac{t\exp\left(\int_0^x [T(y) - R(y) - 2u_0(y)S(y)]\,\mathrm{d}y\right)}{1 + t\int_0^x S(y)\exp\left(\int_0^y [T(s) - R(s) - 2u_0(s)S(s)]\,\mathrm{d}s\right)\mathrm{d}y}. \tag{43}$$

Here $(\psi_0(x)\varphi_0(x))^\top$ and $u_0(x)$ are some special solutions of (39) and (40).

In order to describe the properties of the family of solutions $u^t(x) = \left(\mathcal{B}_t^+ u_0\right)(x)$, $t \in \mathbb{R}$, given by (35) let us state the following:

Proposition 4.

- *Transforms (35) form a one-parameter group*

$$\mathcal{B}^+_{t_1} \circ \mathcal{B}^+_{t_2} = \mathcal{B}^+_{t_1+t_2} \tag{44}$$

which acts transitively on the space of all solutions of the q-difference Riccati equations (6).

- *The solutions* $u^{t_1}(x), u^{t_2}(x), u^{t_3}(x), u^{t_4}(x)$ *satisfy the unharmonical superposition principle*

$$\frac{(u^{t_4}(x) - u^{t_3}(x))(u^{t_1}(x) - u^{t_2}(x))}{(u^{t_3}(x) - u^{t_1}(x))(u^{t_2}(x) - u^{t_4}(x))} = \frac{(t_4 - t_3)(t_1 - t_2)}{(t_3 - t_1)(t_2 - t_4)} \tag{45}$$

for $t_1, t_2, t_3, t_4 \in \mathbb{R}$.

Proof.

- According to proposition 3 any solution of (6) is given by $\mathcal{B}_s u_0$ for some $s \in \mathbb{R}$. Since $\mathcal{B}_{t_1}(\mathcal{B}_t u_0)$ is a solution of (6), we have

$$\mathcal{B}^+_{t_1} \circ \mathcal{B}^+_{t_2} u_0 = \mathcal{B}^+_s u_0. \tag{46}$$

One can find $s \in \mathbb{R}$ by evaluation of both sides of (46) at $x = 0$. Thus, using (33) we obtain

$$t_1 + (t_2 + u_0(0)) = s + u_0(0). \tag{47}$$

- The equality is obtained from (35) by direct calculation. □

The right-hand side of (45) is invariant with respect to the real fractional transformation

$$t_i' = \frac{at_i + b}{ct_i + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \tag{48}$$

of the parameters t_1, t_2, t_3 and t_4. Hence, the left-hand side of (45) is $SL(2, \mathbb{R})$-invariant too.

3. Some integrable cases

In proposition 3 we have constructed the transform which generates the general solution of the q-Riccati equation (6) from a given special one. It was done by the action of a one-parameter group of transformations $\{\mathcal{B}^+_t\}$, $t \in \mathbb{R}$. Formula (35) which defines the group $\{\mathcal{B}^+_t\}_{t\in\mathbb{R}}$ action does depend on the potentials $R(x)$, $S(x)$ and $T(x)$. It does not contain the potential $V(x)$. This allows us to define some method of creation of new integrable systems from a system which one knows how to integrate. In order to do this let us introduce some notation.

By $\mathcal{I}$ we will denote the map

$$\mathcal{I}u(x) := -u(x). \tag{49}$$

The operator which acts on the function $u(x)$ on the right-hand side of the q-Riccati equation (6) will be denoted by $\mathcal{R}_+$, i.e.

$$\mathcal{R}_+u(x) := \partial_q u(x) - T(x)u(x) + R(x)u(qx) + S(x)u(x)u(qx). \tag{50}$$

By $\mathcal{R}_-$ we will denote the operator

$$\mathcal{R}_-u(x) := -\partial_q u(x) + T(x)u(x) - R(x)u(qx) + S(x)u(x)u(qx). \tag{51}$$

It is clear that

$$\mathcal{R}_+ \circ \mathcal{I} = \mathcal{R}_- \qquad \text{and} \qquad \mathcal{R}_- \circ \mathcal{I} = \mathcal{R}_+. \tag{52}$$

Hence, if $u(x)$ is the solution of the equation

$$\mathcal{R}_+ u(x) = V(x) \tag{53}$$

then $\mathcal{I}u(x)$ does satisfy

$$\mathcal{R}_- \circ \mathcal{I}u(x) = V(x) \tag{54}$$

and *vice versa*. The one-parameter groups (35) for equations (53) and (54) will be denoted by $\{\mathcal{B}_t^+\}_{t\in\mathbb{R}}$ and $\{\mathcal{B}_t^-\}_{t\in\mathbb{R}}$ respectively. They are related by

$$\mathcal{B}_t^- = \mathcal{I} \circ \mathcal{B}_t^+ \circ \mathcal{I}. \tag{55}$$

After application of the transform

$$\mathcal{B}_{t_1\cdots t_n}^- := \mathcal{I} \circ \mathcal{B}_{t_1}^+ \circ \mathcal{I} \circ \mathcal{B}_{t_2}^+ \circ \cdots \circ \mathcal{I} \circ \mathcal{B}_{t_n}^+ \circ \mathcal{I} \tag{56}$$

to $u_0(x)$ which one assumes to be the solution of (6) with the potential $V_0(x)$, we obtain the solution

$$u(t_1, \ldots, t_n, x) := \mathcal{B}_{t_1\cdots t_n}^- u_0(x) \tag{57}$$

of (53) with some new potential

$$\mathcal{R}_+ u(t_1, \ldots, t_n, x) = V(t_1, \ldots, t_n, x) \tag{58}$$

which is an n-parameter deformation of the initial one. The same function also satisfies

$$\mathcal{R}_- u(t_1, \ldots, t_n, x) = V(t_2, \ldots, t_n, x). \tag{59}$$

In such a way we obtain the family of q-difference Riccati equations (58) and (59) generated by $u_0(x)$ and the transform (56).

It is worth mentioning here the group-like property of $\mathcal{B}_{t_1\cdots t_n}^-$:

$$\begin{aligned} &\mathcal{B}_{0\ldots 0}^- = \mathrm{id} \\ &\left(\mathcal{B}_{t_1\cdots t_n}^-\right)^{-1} = \mathcal{B}_{-t_n\cdots -t_1}^- \\ &\mathcal{B}_{t_1\cdots t_n}^- \circ \mathcal{B}_{s_1\cdots s_m}^- = \mathcal{B}_{t_1\cdots t_{n-1} t_n+s_1 s_2\cdots s_m}^-. \end{aligned} \tag{60}$$

In the particular case $n = 1$, applying $\{\mathcal{B}_t^-\}_{t\in\mathbb{R}}$ to

$$V_0(x) = -\partial_q u_0(x) + u_0(x) u_0(qx) \tag{61}$$

we obtain

$$\mathcal{R}_+ u(t, x) = V(t, x) \tag{62}$$

$$\mathcal{R}_- u(t, x) = \mathcal{R}_- u_0(x) \tag{63}$$

where $u(t, x) = \mathcal{B}_t^- u_0(x)$. Simple calculation gives a solution

$$u(t, x) = u_0(x) - \frac{t \exp\left(\frac{1}{1-q} \int_0^x \frac{1}{y} \ln \frac{1+(1-q) y u_0(y)}{1-(1-q) y u_0(qy)} \, \mathrm{d}_q y\right)}{1 + t \int_0^x \frac{1}{1+(1-q) y u_0(y)} \exp\left(\frac{1}{1-q} \int_0^y \frac{1}{s} \ln \frac{1+(1-q) s u_0(s)}{1-(1-q) s u_0(qs)} \, \mathrm{d}_q s\right) \mathrm{d}_q y} \tag{64}$$

for the potential

$$\begin{aligned} V(t, x) = V_0(x) &+ 2\partial_q u_0(x) \\ &- 2\partial_q \frac{t \exp\left(\frac{1}{1-q} \int_0^x \frac{1}{y} \ln \frac{1+(1-q) y u_0(y)}{1-(1-q) y u_0(qy)} \, \mathrm{d}_q y\right)}{1 + t \int_0^x \frac{1}{1+(1-q) y u_0(y)} \exp\left(\frac{1}{1-q} \int_0^y \frac{1}{s} \ln \frac{1+(1-q) s u_0(s)}{1-(1-q) s u_0(qs)} \, \mathrm{d}_q s\right) \mathrm{d}_q y}. \end{aligned} \tag{65}$$

In the limit of $q \to 1$ equations (62) and (63) correspond to the proper differential equations and (64) tends to the already known [Mi] form of the solution

$$u(t,x) = u_0(x) - \frac{\partial}{\partial x} \ln \left(1 + t \int_0^x \exp\left(2 \int_0^y u_0(z)\,\mathrm{d}z \right) \mathrm{d}y \right) \tag{66}$$

of (3) with potential given by

$$V(t,x) = V_0(x) + 2\frac{\partial}{\partial x} u_0(x) - 2\frac{\partial^2}{\partial x^2} \ln \left(1 + t \int_0^x \exp\left(2 \int_0^y u_0(z)\,\mathrm{d}z \right) \mathrm{d}y \right). \tag{67}$$

Combining (62) and (63) we find the formula

$$u(t,x) = \tfrac{1}{2}\Big\{ (1-q)x\tfrac{1}{2}(V(t,x) - \mathcal{R}_- u_0(x)) \pm \sqrt{\left[(1-q)x\tfrac{1}{2}(V(t,x) - \mathcal{R}_- u_0(x))\right]^2 + 2(V(t,x) + \mathcal{R}_- u_0(x))} \Big\} \tag{68}$$

which expresses the solution $u(t,x)$ by the t-deformed potential $V(t,x)$ and the initial solution $u_0(x)$.

As an example we shall apply our method in the case when the initial potential and corresponding solution for the chain of q-Riccati equation (58) are given by

$$V_0(x) = -a\frac{1-q^\alpha}{1-q} x^{\alpha-1} + a^2 q^\alpha x^{2\alpha} \tag{69}$$

and

$$u_0(x) = a x^\alpha \tag{70}$$

respectively. Using formulae (65) and (64) we find that

$$\begin{aligned} V(t,x) &= a\frac{1-q^\alpha}{1-q} x^{\alpha-1} + a^2 q^\alpha x^{2\alpha} \\ &- 2t \frac{\frac{\exp_R(aq^{\alpha+1}x^{\alpha+1})}{\exp_R(-aq^\alpha x^{\alpha+1})}}{\left(1 + t\int_0^x \frac{\exp_R(aq^{\alpha+1}y^{\alpha+1})}{\exp_R(-aq^\alpha y^{\alpha+1})}\,\mathrm{d}_q y\right)\left(1 + qt\int_0^x \frac{\exp_R(aq^{2(\alpha+1)}y^{\alpha+1})}{\exp_R(-aq^{2\alpha+1}y^{\alpha+1})}\,\mathrm{d}_q y\right)} \\ &\times \left\{ (1+q^\alpha)aq^{\alpha+1}x^\alpha \left(1 + t\int_0^x \frac{\exp_R(aq^{\alpha+1}y^{\alpha+1})}{\exp_R(-aq^\alpha y^{\alpha+1})}\,\mathrm{d}_q y\right) - t\frac{\exp_R(aq^{\alpha+1}x^{\alpha+1})}{\exp_R(-aq^{2\alpha+1}x^{\alpha+1})} \right\} \end{aligned} \tag{71}$$

and

$$u(t,x) = ax^\alpha - \frac{t\frac{\exp_R(ax^{\alpha+1})}{\exp_R(-aq^\alpha x^{\alpha+1})}}{1 + t\int_0^x \frac{\exp_R(aq^{\alpha+1}y^{\alpha+1})}{\exp_R(-aq^\alpha y^{\alpha+1})}\,\mathrm{d}_q y} \tag{72}$$

where

$$\exp_R(x) = \sum_{n=0}^{\infty} \frac{1}{R(q)\cdots R(q^n)} x^n \tag{73}$$

is the generalized exponential function in the sense of [O] with

$$R(x) = \frac{1 - x^{\alpha+1}}{(1-q)x^{\alpha+1}}. \tag{74}$$

One may of course continue applying the procedure step by step. We will not pursue this way here since the expressions for successive potentials and solutions are very complicated and not illuminating.

In the limit $q \to 1$, formulae (72) and (71) tend to the form

$$u(t,x) = ax^{\alpha} - \frac{t\exp\left(\frac{2a}{\alpha+1}x^{\alpha+1}\right)}{1 + t(\alpha+1)\left(-\frac{\alpha+1}{2a}\right)^{\frac{\alpha}{2(\alpha+1)}}\mathcal{A}(x)} \tag{75}$$

$$\begin{aligned} V(t,x) &= a\alpha x^{\alpha-1} + a^2x^{2\alpha} - 4t\frac{ax^{\alpha}\exp\left(\frac{2a}{\alpha+1}x^{\alpha+1}\right)}{1 + t(\alpha+1)\left(-\frac{\alpha+1}{2a}\right)^{\frac{1}{\alpha+1}}\mathcal{A}(x)} \\ &\quad + 2t^2\frac{\exp^2\left(\frac{2a}{\alpha+1}x^{\alpha+1}\right)}{\left(1 + t(\alpha+1)\left(-\frac{\alpha+1}{2a}\right)^{\frac{1}{\alpha+1}}\mathcal{A}(x)\right)^2} \end{aligned} \tag{76}$$

where the function $\mathcal{A}(x)$ is given by

$$\begin{aligned} \mathcal{A}(x) &= \Gamma\left(\frac{1}{\alpha+1}\right) - \left(-\frac{\alpha+1}{2a}\right)^{\frac{\alpha}{2(\alpha+1)}} x^{-\frac{\alpha}{2}}\exp\left(\frac{\alpha}{\alpha+1}x^{\alpha+1}\right) \\ &\quad \times \mathcal{W}_{-\frac{\alpha}{2(\alpha+1)},\frac{1}{2}-\frac{\alpha}{2(\alpha+1)}}\left(-\frac{2a}{\alpha+1}x^{\alpha+1}\right). \end{aligned} \tag{77}$$

Here $\Gamma(x)$ is the gamma function and $\mathcal{W}(x)$ is the Whittaker function [Ry-G].

Let us consider the subcase of $\alpha = 0$, corresponding to the constant potential

$$V_0(x) = a^2. \tag{78}$$

One-parameter deformation (71) of the above (78) is given in this case by the family of potentials

$$V(t,x) = \frac{a^2\left(1-\frac{t}{2a}\right)^2\exp(-2ax) - 3at\left(1-\frac{t}{2a}\right) + \frac{t^2}{4}\exp(2ax)}{\left(\left(1-\frac{t}{2a}\right)\exp(-ax) + \frac{t}{2a}\exp(ax)\right)^2} \tag{79}$$

and solution (72) of the differential Riccati equation with this potential reads

$$u(t,x) = a - \frac{t}{\left(1-\frac{t}{2a}\right)\exp(-2ax) + \frac{t}{2a}}. \tag{80}$$

In the special case $t = a$, the potential (79) reduces to the Rosen–Morse potential

$$V(a,x) = a^2 - \frac{2a^2}{\cosh^2 ax}. \tag{81}$$

Conversely, transforming the variable by translation $x \to x - \frac{1}{2a}\ln\left(\frac{2a}{t} - 1\right)$ from (81), one obtains (79). The two-parameter deformation $\mathcal{B}^{-}_{t_1t}$ of (78) generates the potential

$$\begin{aligned} V(t_1,t,x) &= a^2 + \frac{4at\left(1-\frac{t}{2a}\right)}{\left(\left(1-\frac{t}{2a}\right)\exp(-ax) + \frac{t}{2a}\exp(ax)\right)^2} \\ &- 2\frac{\exp(-2ax)\left[-2at_1\left(1-\frac{t}{2a}\right)^2\left(1+\frac{t_1}{2a}-\frac{t_1t}{2a^2}\right) - 2\frac{t_1^2t}{a}\left(1-\frac{t}{2a}\right)^3\right]}{\left(1+\frac{t_1}{2a}-\frac{t_1t}{2a^2}-\frac{t_1}{2a}\left(1-\frac{t}{2a}\right)^2\exp(-2ax) + \frac{t_1t}{a}\left(1-\frac{t}{2a}\right)x + \frac{t_1t^2}{8a^3}\exp(2ax)\right)^2} \\ &+ \frac{\left[\frac{t_1^2t^2}{a^2}\left(1-\frac{t}{2a}\right)^2\right] + \exp(2ax)\left[\frac{t_1t^2}{2a}\left(1+\frac{t_1}{2a}-\frac{t_1t}{2a^2}\right) - \frac{t_1^2t^3}{2a^3}\left(1-\frac{t}{2a}\right)\right]}{\left(1+\frac{t_1}{2a}-\frac{t_1t}{2a^2}-\frac{t_1}{2a}\left(1-\frac{t}{2a}\right)^2\exp(-2ax) + \frac{t_1t}{a}\left(1-\frac{t}{2a}\right)x + \frac{t_1t^2}{8a^3}\exp(2ax)\right)^2} \\ &+ \frac{x\exp(-2ax)\left[-2t_1t\left(1-\frac{t}{2a}\right)\right] + x\exp(2ax)\left[\frac{t_1^2t^3}{2a^2}\left(1-\frac{t}{2a}\right)\right]}{\left(1+\frac{t_1}{2a}-\frac{t_1t}{2a^2}-\frac{t_1}{2a}\left(1-\frac{t}{2a}\right)^2\exp(-2ax) + \frac{t_1t}{a}\left(1-\frac{t}{2a}\right)x + \frac{t_1t^2}{8a^3}\exp(2ax)\right)^2} \end{aligned} \tag{82}$$

and the solution of the difference Riccati equation (3) for this potential is

$$u(t_1, t, x) = -a + \frac{t}{\left(1 - \frac{t}{2a}\right)\exp(-2ax) + \frac{t}{2a}}$$
$$- \frac{t_1\left(1 - \frac{t}{2a}\right)^2 \exp(-2ax) + \frac{t_1 t}{a}\left(1 - \frac{t}{2a}\right) + \frac{t_1 t^2}{4a^2}\exp(2ax)}{1 + \frac{t_1}{2a} - \frac{t_1 t}{2a^2} - \frac{t_1}{2a}\left(1 - \frac{t}{2a}\right)^2 \exp(-2ax) + \frac{t_1 t}{a}\left(1 - \frac{t}{2a}\right)x + \frac{t_1 t^2}{8a^3}\exp(2ax)}. \tag{83}$$

The potentials (79), (81) and (82) are known and were obtained by different methods, for example in [I-H, Mi, M-S, L-R, St, Sta, M, Ma].

Acknowledgments

We would like to thank the referees for careful reading of the manuscript and suggestions which made the paper more readable. This work is supported in part by KBN grant 2 PO3 A 012 19.

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